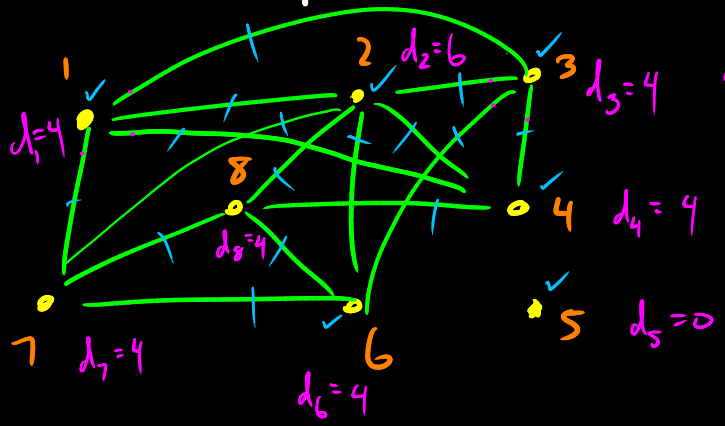
Last Time: Graph Theory !!

Defn: (Simple) graph $G = (V(G), E(G))$ satisfies

$V(G)$ is a set and $E(G) \subseteq \{\{u,v\} : u,v \in V(G), u \neq v\}$

(Pictures in the plane $\longleftrightarrow$ simple graphs).

Ex:



$d_v = \deg(v)$ is in pink.

$= G$

nice convention to save space

$\{1,8\}$ " = " 18

i.e. $\{u,v\} \rightsquigarrow uv$

$\star V(G) = \{1,2,3,4,5,6,7,8\}$ $\longleftarrow$

$\star E(G) = \{12, 13, 14, 17, 23, 24, 26, 27, 28, 34, 36, 48, 67, 68, 78\}$.

NB: $uv = \{u,v\} = \{v,u\} = vu$, so edges are "directionless".

Defn: $v \in V(G)$ has neighborhood $N_G(v) := \{u : uv \in E(G)\}$.

The degree of $v \in V(G)$ is $\deg_G(v) = \# N_G(v)$.

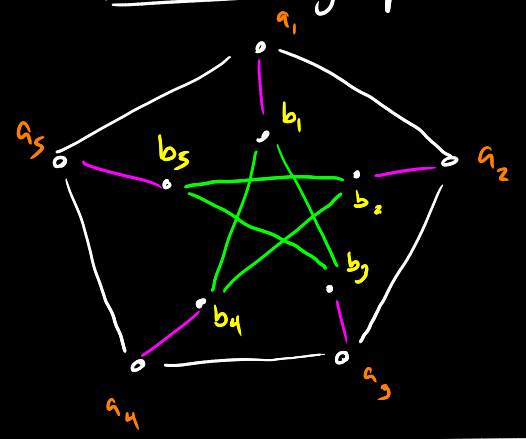
Ex: In G above, we have $\sum_{v \in V(G)} \deg(v) = 4+4+4+4+4+4+6 = 30$

(NB: $4 \leq 4 \leq 4 \leq 4 \leq 4 \leq 4 \leq 6$ is the degree sequence of G.)

On the other hand, $\# E(G) = 15$ $\longleftarrow$

Ex: In $K_n = \left([n], \left[\{ij : i,j \in [n], i \neq j\}\right]_n\right)$, $\binom{[n]}{2}$, we have for all v, $\deg_{K_n}(v) = n-1$, so $\sum_{i=1}^{n} \deg_{K_n}(i) = \sum_{i=1}^{n} (n-1) = n(n-1)$. On the other hand, $\# E(K_n) = \# \binom{[n]}{2} = \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)(n-2)!}{2(n-2)!} = \frac{n(n-1)}{2}$.

Ex: The Petersen graph is depicted below:

$P =$ 

$\deg(v) = 3$, so $\sum_{v \in V(P)} \deg(v) = 3 \cdot 10 = 30$

while $\#E(P) = 3 \cdot 5 = 15$.

So again $\sum_{v \in V(P)} \deg(v) = 2\,\#E(P)$. ✓

Prop (Handshaking Lemma): Let G be a graph. Then

$$\sum_{v \in V(G)} \deg_G(v) = 2\,\#E(G).$$

pf: Let G be a graph. Define

$$I = \{(v,e) \in V(G) \times E(G) : v \text{ is incident to } e\}.$$

We count I in two ways.

Count 1: Express $I = \bigcup_{v \in V(G)} \{(v,e) : e \in E(G)\}$.

Letting $I_v := \{(v,e) : e \in E(G)\}$ for each $v \in V(G)$, we have $I = \bigcup_{v \in V(G)} I_v$ and if $(u,e) \in I_v \cap I_w$ for some vertices u, v, w, we have $(v,e') = (u,e) = (w,e'')$ for some $e', e'' \in E(G)$; Then $v = u = w$. Hence $\{I_v : v \in V(G)\}$ is a pairwise disjoint collection of subsets of I. By the Addition Principle, $\#I = \#\bigcup_{v \in V(G)} I_v = \sum_{v \in V(G)} \#I_v$. For each $v \in V(G)$, $f_v : I_v \to \{e \in E(G) : v \text{ is an end of } e\}$ defined by $f_v(v,e) = e$ is a bijection (check ✓).

Hence $\#I_v = \#\{e \in E(G) : v \text{ is an end of } e\} = \deg(v)$.

Hence $\#I = \sum_{v \in V(G)} \deg_G(v)$.

Count 2: We write for each $e \in E(G)$

$$J_e := \{(v,e) \in I : v \text{ is an end of } e\}.$$

For each $e \in E(G)$ we have $\#J_e = 2$.

Indeed, if e has ends u, v, then $J_e = \{(u,e), (v,e)\}$ (because every edge has two ends ...). In particular, expressing

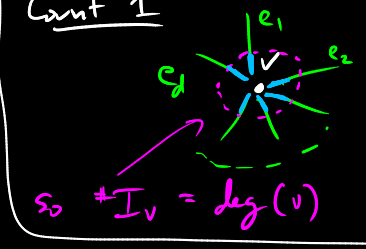
$$I = \bigcup_{e \in E(G)} J_e$$

and noting $\{J_e : e \in E(G)\}$ is a pairwise disjoint collection (verify!), we have

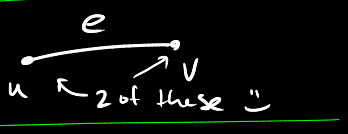
$$\#I = \#\bigcup_{e \in E(G)} J_e = \sum_{e \in E(G)} \#J_e = \sum_{e \in E(G)} 2 = 2\#E(G)$$

by the Addition Principle

Count 1



Count 2:



Comparing our counts we see $\sum_{v \in V(G)} \deg(v) = \#I = 2\#E(G)$. ☑

Remark: The name "handshaking lemma" comes from this interpretation: Let H be a set of individuals, and let A be the set of pairs of people who are mutual friends. If everyone in H attends a party and each person shakes the hand of each of their friends, $\#E(G)$ counts the number of handshakes. However, $\sum_{v \in V(G)} \deg(v)$ counts each handshake once from the perspective of each person... so it double-counts each handshake. (where $G = (H, A)$). ☑

Cor: Suppose G has an odd number of vertices. Then G has a vertex of even degree.

Pf: By the Handshake Lemma, $\sum_{v \in V(G)} \deg(v) = 2\#E(G)$, so $\sum_{v \in V(G)} \deg(v)$ is even. But the sum of oddly many odd integers is odd, so at least one vertex must have even degree as there are oddly many. ☑

Another Use of the Handshake Lemma: Rule out some sequences of nonnegative integers as degree sequences.

Ex: Is $1 \le 1 \le \boxed{2} \le 3 \le 3 \le 5$ a degree sequence?

No! If there were such a G, then

$$\#E(G) = \tfrac{1}{2}(1+1+2+3+3+5) = \tfrac{1}{2}(15) = \tfrac{15}{2} = 7+\tfrac{1}{2} \notin \mathbb{N}_0,$$

which is impossible as a cardinality! ☐

Exercise: Prove every graph has evenly many vertices of odd degree...
↳ Small generalization of the preceding corollary...

Ex: Is $1 \le 1 \le 1 \le 3 \le \boxed{6}$ a degree sequence of a simple graph?
↳ No, but for non-handshaking reasons... ← try it...

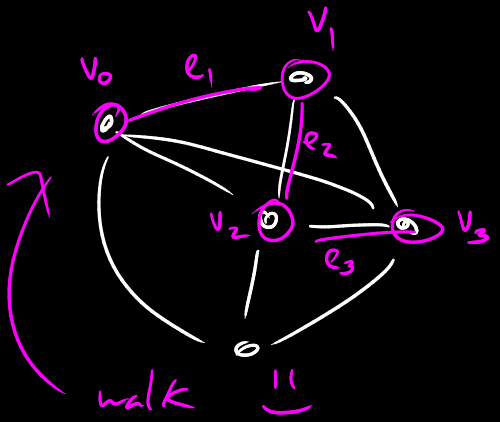
Ex: Is $1 \le 1 \le 1 \le 3 \le 4$ a degree sequence of a simple graph?
↳ Try it: Either build a graph w/ this degree sequence, or obtain a contradiction from such a "graph".

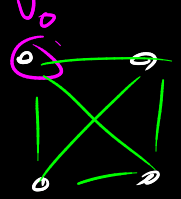
Defn: A walk in graph G is a sequence

$$(v_0, e_1, v_1, e_1, v_2, e_2, \dots, v_{n-1}, e_n, v_n)$$ satisfying:

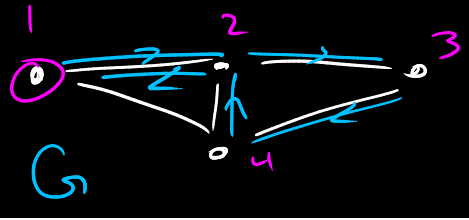
1. $v_i \in V(G)$ for all $0 \le i \le n$.
2. $e_i \in E(G)$ for all $1 \le i \le n$.
3. the ends of e_i are v_{i-1} and v_i for all $1 \le i \le n$.

Pictures:





(v_0)



$(1, 12, 2, 23, 3, 34, 4, 24, 2, 12, 1)$ is a perfectly good walk ☐